

# Clique Separator Decomposition of Hole- and Diamond-Free Graphs and Algorithmic Consequences

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## Abstract

Clique separator decomposition introduced by Tarjan and Whitesides is one of the most important graph decompositions. A graph is an *atom* if it has no clique separator. A *hole* is a chordless cycle with at least five vertices, and an *antihole* is the complement graph of a hole. A graph is *weakly chordal* if it is hole- and antihole-free.  $K_4 - e$  is also called *diamond*. *Paraglider* has five vertices four of which induce a diamond, and the fifth vertex sees exactly the two vertices of degree two in the diamond. In this paper we show that atoms of hole- and diamond-free graphs (of hole- and paraglider-free graphs, respectively) are either weakly chordal or of a very specific structure. Hole- and paraglider-free graphs are perfect graphs. The structure of their atoms leads to efficient algorithms for various problems.

**Keywords:** Clique separator decomposition; hole- and diamond-free graphs; hole- and paraglider-free graphs; perfect graphs; efficient algorithms.

## 1 Introduction, Motivation and Related Work

A *clique separator* (or *clique cutset*) of a graph  $G$  is a clique  $K$  such that  $G[V \setminus K]$  has more connected components than  $G$ . An *atom* is a graph without clique separator. In [32, 34], it is shown that a clique separator decomposition tree of a graph can be determined in polynomial time, and in [32], this decomposition is applied to various problems such as Minimum Fill-in, Maximum Weight Independent Set (MWIS), Maximum Weight Clique and Coloring; if the problem is solvable in polynomial time on the atoms of a hereditary graph class  $\mathcal{C}$ , it is solvable in polynomial time on class  $\mathcal{C}$ . In this paper, we are going to analyze the structure of atoms in two subclasses of hole-free graphs.

A *hole* is a chordless cycle with at least five vertices, and an *antihole* is the complement graph of a hole. A graph is *hole-free* (*antihole-free*, respectively) if it contains no induced subgraph which is isomorphic to a hole (an antihole, respectively).

$K_4 - e$  (i.e., a clique of four vertices minus one edge) is called *diamond*. A *paraglider* has five vertices four of which induce a diamond, and the fifth vertex sees exactly the two vertices of degree two in the diamond (see Figure 1). Note that paraglider is the

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complement graph of the disjoint union  $P_2 \cup P_3$  (where  $P_n$  denotes a chordless path with  $n$  vertices and  $n - 1$  edges).

Cycle properties of graphs and their algorithmic aspects play a fundamental role in combinatorial optimization, discrete mathematics and computer science. Various graph classes are characterized in terms of cycle properties - among them are the classes of chordal graphs, weakly chordal graphs and perfect graphs which are of fundamental importance for algorithmic graph theory and various applications. A graph is *chordal* (also called *triangulated*) if it is hole- and  $C_4$ -free (where  $C_4$  denotes the chordless cycle of four vertices). See e.g. [13, 22, 30] for the many facets of chordal graphs. A graph is completely decomposable by clique separator decomposition if and only if it is chordal. A graph is *weakly chordal* (also called *weakly triangulated*) if it is hole- and antihole-free. These graphs have been extensively studied in [25, 26, 28, 31]; they are perfect. In [2, 27], recognition of weakly chordal graphs is solved in time  $\mathcal{O}(m^2)$ , and the MWIS problem on weakly chordal graphs is solved in time  $\mathcal{O}(n^4)$ . Chordal graphs are weakly chordal.

The celebrated *Strong Perfect Graph Theorem* (SPGT) by Chudnovsky et al. says:

**Theorem 1** (SPGT [19]). *A graph is perfect if and only if it is odd-hole-free and odd-antihole-free.*

It is also well known that a graph is the line graph of a bipartite graph if and only if it is (claw,diamond,odd-hole)-free (see e.g. [13]). These graphs play a fundamental role in the proof of the SPGT.

Since every hole  $C_k$ ,  $k \geq 7$ , contains the disjoint union of  $P_2$  and  $P_3$  (and the paraglider is the complement graph of  $P_2 \cup P_3$ ), it follows that HP-free graphs are  $\overline{C_k}$ -free for every  $k \geq 7$ . Thus, by the SPGT, HP-free graphs are perfect. Our structural results for atoms of HP-free graphs, however, give a more direct way to show perfection of HP-free graphs.

Hole- and diamond-free graphs generalize the important class of chordal bipartite graphs (which are exactly the hole- and triangle-free graphs), and diamond-free chordal graphs are the well-known block graphs - see [13] for various characterizations and the importance of chordal bipartite graphs as well as of block graphs. In [10, 17], various characterizations of (dart,gem)-free chordal graphs are given; among others, it is shown that a graph is (dart,gem)-free chordal if and only if it results from substituting cliques into the vertices of a block graph.

Recently there has been much work on related classes such as even-hole-free (forbidding also  $C_4$ ) and diamond-free graphs [29] (see also [33]) and [21] dealing with the structure and recognition of  $C_4$ - and diamond-free graphs.

Hole- and paraglider-free graphs obviously generalize chordal graphs. The classes of weakly chordal graphs and HP-free graphs are incomparable as the examples of paraglider (which is weakly chordal but not HP-free) and  $\overline{C_6}$  (which is HP-free but not weakly chordal) show but HP-free graphs are closely related to weakly chordal graphs:

Our main result in this paper shows that atoms of hole- and paraglider-free graphs (HP-free graphs for short) are either weakly chordal or of a very simple structure close to matched co-bipartite graphs. By [32], this has various algorithmic consequences; in section 5, we describe these and others.

## 2 Further Basic Notions

Let  $G$  be a graph with vertex set  $V(G) = V$  and edge set  $E(G) = E$ . Adjacency of vertices  $x, y \in V$  is denoted by  $xy \in E$ , or  $x \sim y$ , or we simply say that  $x$  and  $y$  see each other. Nonadjacency is denoted by  $xy \notin E$ , or  $x \not\sim y$ , or  $x$  and  $y$  miss each other.

The *open neighborhood*  $N(v)$  of a vertex  $v$  in  $G$  is  $N(v) = \{u \mid uv \in E\}$ , the *closed neighborhood* of  $v$  is  $N[v] = N(v) \cup \{v\}$ , and the *antineighborhood*  $A(v)$  of  $v$  is  $A(v) = \{u \mid u \neq v \text{ and } uv \notin E\}$ .

The neighborhood  $N(X)$  of a subset  $X \subseteq V$  is the set of all neighbors of  $x \in X$  outside  $X$ . For a subgraph  $H$  of  $G$ , let  $N_H(x)$  denote the set  $N(x) \cap V(H)$  and let  $N_H(X)$  denote the set  $N(X) \cap V(H)$ .

For graph  $G$ , let  $\overline{G}$  (or  $\text{co-}G$ ) denote the complement graph of  $G$ , i.e.,  $\overline{G} = (V(G), \{xy \mid x \neq y \text{ and } x \not\sim y\})$ . For  $H \subseteq V$ , let  $G[H]$  denote the induced subgraph of  $H$  in  $G$ .

Let  $P_k$  denote a chordless path with  $k$  vertices  $x_1, \dots, x_k$  and edges  $x_i x_{i+1}$ ,  $1 \leq i \leq k-1$ , and let  $C_k$  denote a chordless cycle with the same  $k$  vertices and edges  $x_i x_{i+1}$ ,  $1 \leq i \leq k-1$ , and  $x_k x_1$ .

A vertex set  $U \subseteq V$  is *independent* if the vertices of  $U$  are pairwise nonadjacent.  $U$  is a *clique* if the vertices of  $U$  are pairwise adjacent. Let  $S_r$  ( $K_r$ , respectively) denote an independent vertex set (a clique, respectively) with  $r$  vertices.

For vertex  $x$  of graph  $G$  and  $H \subseteq V(G)$ ,  $x \textcircled{1} H$  means that  $x$  is adjacent to all vertices of  $H$ . In this case, we also say that  $x$  is *total* or *universal* with respect to  $H$ . Correspondingly,  $x \textcircled{0} H$  means that  $x$  is adjacent to no vertex of  $H$ .

For  $H \subseteq V(G)$  and  $Q \subseteq V(G)$  with  $H \cap Q = \emptyset$ ,  $H \textcircled{1} Q$  means that every vertex of  $H$  is adjacent to every vertex of  $Q$  (we also say that  $H$  and  $Q$  form a *join*) and  $H \textcircled{0} Q$  means that no vertex of  $H$  is adjacent to any vertex of  $Q$  ( $H$  and  $Q$  form a *co-join*).

Let  $G$  be a graph.  $G \setminus H$  or  $G - H$  denotes the graph  $G[V(G) - V(H)]$  induced by the set of vertices  $V(G) - V(H)$ .

Let  $\mathcal{F}$  be a set of graphs.  $G$  is  $\mathcal{F}$ -free if no induced subgraph of  $G$  is an element of  $\mathcal{F}$ . As already mentioned,  $G$  is *hole-free* (is *antihole-free*, respectively) if no induced subgraph of  $G$  is isomorphic to a hole (an antihole, respectively).

A *co-matched bipartite graph* results from a complete bipartite graph  $K_{k,k}$  by deleting a perfect matching. A *matched co-bipartite graph* is the complement of a co-matched bipartite graph, i.e., it consists of two disjoint cliques of the same size  $k$ , and the edges between them form a matching with  $k$  edges.

Note that  $\overline{C_6}$  is a matched co-bipartite graph with six vertices. Let  $A$  be a matched co-bipartite graph. Then  $\text{left}(A)$  denotes one of the maximal cliques of  $A$  and  $\text{right}(A)$  denotes the other maximal clique of  $A$ . Clearly  $\text{left}(A)$  and  $\text{right}(A)$  form a bipartition of the co-matched bipartite graph  $\overline{A}$  (and thus a corresponding partition of the vertex set of  $A$ ). Subsequently, the edges between  $\text{left}(A)$  and  $\text{right}(A)$  are called *matching edges*.

## 3 Adjacency Properties for (Hole,Paraglider)-Free Graphs Containing $\overline{C_6}$

In this section we describe some adjacency properties of HP-free graphs containing  $\overline{C_6}$  which will be useful in the structural description of atoms of hole- and paraglider-free

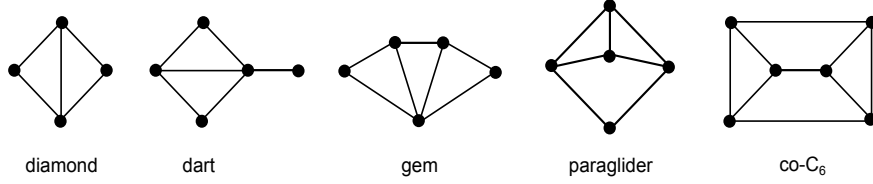


Figure 1: diamond, dart, gem, paraglider, and co- $C_6$ .

graphs.

### 3.1 Neighbors of $\overline{C_6}$ in HP-Free Graphs

Throughout this section, let  $G$  be an HP-free graph. As mentioned already in the introduction, the only possible antihole in an HP-free graph is  $\overline{C_6}$ ; if  $G$  is  $\overline{C_6}$ -free, it is weakly chordal. The following propositions are dealing with HP-free graphs containing  $\overline{C_6}$ . Obviously, the following holds:

**Proposition 1.** *Pairs  $x, y$  with  $x \not\sim y$  in a  $\overline{C_6}$   $A$  are endpoints of a  $P_4$   $(x, a, b, y)$  and two  $P_3$ 's  $(x, c, y)$ ,  $(x, d, y)$  such that  $(c, a, b, d)$  is another  $P_4$  in  $A$ .*

Let  $A$  be a graph isomorphic to a  $\overline{C_6}$ . The set of vertices outside  $A$  having distance  $i \geq 1$  from  $A$  will be denoted by  $D_i(A)$ . Moreover,  $D_1 = D_1(A) = A_1 \cup \dots \cup A_6$ , where  $A_i$ ,  $i \in \{1, \dots, 6\}$ , denotes the set of vertices outside  $A$  with distance one from  $A$  and having exactly  $i$  neighbors in  $A$  (note that  $A_i$  contain only vertices which are not in  $A$ ).

Obviously, the next property holds:

**Proposition 2.** *If  $x, y \in A_1$  with  $x \sim y$ , and  $N_A(x) = \{t\}$ ,  $N_A(y) = \{z\}$  with  $t \neq z$  then  $t \sim z$ .*

For neighbors outside  $A$  which see more than one vertex in  $A$ , the situation is as follows:

**Proposition 3.**

- (i) *The two  $A$ -neighbors of any vertex in  $A_2$  form an edge in  $A$ .*
- (ii) *The three  $A$ -neighbors of any vertex in  $A_3$  form a triangle in  $A$ .*
- (iii)  $A_4 = A_5 = \emptyset$ .
- (iv)  $A_6$  *is a clique. Moreover, in a hole- and diamond-free graph,  $A_6 = \emptyset$ .*
- (v) *If  $x$  sees  $A$  and  $N_A(x)$  is not a clique then  $x \in A_6$ .*

**Proof.** (i): If  $x \in A_2$  sees  $y$  and  $z$  in  $A$  with  $y \not\sim z$  then by Proposition 1, there is a  $P_4$   $P$  in  $A$  with endpoints  $y$  and  $z$ . It follows that  $x$  together with  $P$  induce a  $C_5$  in  $G$ , a contradiction.

(ii): If the neighborhood of  $x \in A_3$  in  $A$  is not a triangle then without loss of generality,  $x$  sees two vertices in  $\text{left}(A)$ , say  $a$  and  $b$ , and one in  $\text{right}(A)$ , say  $c$ . If  $c$  misses  $a$  and  $b$  then

$x, a, b, c$  together with the neighbor of  $c$  in  $\text{left}(A)$  induce a paraglider, and if  $c$  sees  $a$  then  $x, a, b, c$  together with the neighbor of  $b$  in  $\text{right}(A)$  induce a paraglider - contradiction.

(iii): If  $x \in A_4$  sees all three vertices in  $\text{left}(A)$ , say  $a, b, c$ , and one in  $\text{right}(A)$ , say  $d$ , then if  $a$  sees  $d$ ,  $x, a, b, d$  together with the neighbor of  $b$  in  $\text{right}(A)$  induce a paraglider. If  $x$  sees two vertices in  $\text{left}(A)$ , say  $a, b$ , and two vertices in  $\text{right}(A)$ , say  $c, d$  then if  $a$  sees  $c$  and  $b$  sees  $d$ ,  $x, a, b, c$  and the matching edge which  $x$  is missing induce a  $C_5$ . If  $a$  misses  $d$  and  $b$  sees  $c$  then  $x, a, b, c$  and the neighbor of  $a$  in  $\text{right}(A)$  induce a paraglider.

If  $x \in A_5$  sees all three vertices in  $\text{left}(A)$  and two in  $\text{right}(A)$ , say  $d, e$ , then  $x, d, e$  together with the vertex  $f$  which  $x$  misses in  $\text{right}(A)$  and the neighbor of  $f$  in  $\text{left}(A)$  induce a paraglider.

(iv): If there are  $x, y \in A_6$  with  $x \not\sim y$  then  $x$  and  $y$  together with any  $P_1 \cup P_2$  from  $A$  form a paraglider. Moreover, the vertices of any  $P_3$  in  $A$  together with any vertex of  $A_6$  induce a diamond.

(v): This property easily follows from the preceding ones.  $\square$

**Proposition 4.** *Let  $x \sim y$ . If  $x \in A_1$  and  $y \in A_2 \cup A_3$  or  $x \in A_2$  and  $y \in A_3$  then  $N_A(x)$  and  $N_A(y)$  are comparable with respect to set inclusion.*

**Proof.** As before, let  $A$  be a  $\overline{C_6}$ , say with cliques  $\text{left}(A) = \{v_1, v_2, v_3\}$ ,  $\text{right}(A) = \{v_4, v_5, v_6\}$  and matching edges  $v_1v_4, v_2v_5$  and  $v_3v_6$ .

First let  $x \in A_1$ ; without loss of generality, let  $N_A(x) = \{v_1\}$  and assume that  $y \not\sim v_1$ . Recall that  $y \in A_2$  or  $y \in A_3$ . If  $\{v_2, v_3\} \subseteq N_A(y)$  then  $x, y, v_1, v_2, v_3$  induce a paraglider. Thus  $y$  must see at least one vertex from  $\text{right}(A)$ . If  $y$  sees  $v_5$  then either  $x, v_1, v_4, v_5, y$  or  $x, v_1, v_2, v_5, y$  is a  $C_5$  since by Proposition 3,  $N_A(y)$  is a clique, and similarly if  $y$  sees  $v_6$ . Thus  $y$  misses  $v_5$  and  $v_6$  which implies that  $y$  sees  $v_4$ . Since by assumption,  $y$  misses  $v_1$ ,  $y$  sees  $v_4$  and  $v_2$  or  $v_3$  but this contradicts Proposition 3.

Now let  $x \in A_2$  and  $y \in A_3$ ; by Proposition 3,  $N_A(y) = \text{left}(A)$  or  $N_A(y) = \text{right}(A)$  and  $N_A(x)$  is an edge in  $A$ . If  $N_A(x) = \{v_1, v_2\}$  and  $N_A(x)$  and  $N_A(y)$  are not comparable then  $N_A(y) = \text{right}(A)$  but now  $x, v_2, v_3, v_6, y$  is a  $C_5$  - contradiction. If however  $N_A(x) = \{v_1, v_4\}$  and without loss of generality,  $N_A(y) = \text{left}(A)$  then  $x, y, v_2, v_5, v_4$  is a  $C_5$  which shows Proposition 4.  $\square$

**Proposition 5.** *For all  $x, y \in A_2$  with  $x \sim y$ ,  $N_A(x) \cup N_A(y)$  is a clique.*

**Proof.** By Proposition 3,  $N_A(x)$  and  $N_A(y)$  are edges. Assume to the contrary that there are  $z \in N_A(x)$  and  $t \in N_A(y)$  with  $z \not\sim t$ . Thus  $z \notin N_A(y)$  and  $t \notin N_A(x)$ . By Proposition 1, there is a  $P_4(z, u, v, t)$  in  $A$ . Since  $N_A(x)$  is an edge,  $x$  misses  $v$ , and likewise  $y$  misses  $u$ . To avoid a hole in the subgraph induced by  $\{x, z, u, v, t, y\}$ , we obtain  $x \sim u$  and  $y \sim v$  which implies that  $N_A(x) \cup N_A(y) = \{z, u, v, t\}$ . Then by Proposition 1 there is a  $P_3(z, w, t)$  in  $A$  such that  $x$  and  $y$  miss  $w$  and consequently  $x, z, w, t, y$  induce a  $C_5$  in  $G$ , a contradiction.  $\square$

Now it is easy to see that by Propositions 2, 3, 4, and 5, we obtain:

**Corollary 1.** *For all  $x, y \in D_1$  with  $x \sim y$  and at least one of  $x, y$  does not belong to  $A_3$ ,  $N_A(x) \cup N_A(y)$  is a clique.*

**Proposition 6.** *Let  $x, y \in D_1$  with  $x \not\sim y$  be the endpoints of a chordless path  $P$  whose internal vertices do not belong to  $D_1 \cup A$ . Then*

- (i)  *$P$  contains exactly three vertices  $x, w, y$  and*
- (ii)  *$N_A(x)$  and  $N_A(y)$  are comparable.*

**Proof.** (i): Assume to the contrary that  $P$  contains at least four vertices. Let  $u$  and  $v$  be two vertices of  $A$  such that  $u \in N_A(x)$  and  $v \in N_A(y)$  and let  $Q$  be a chordless path in  $A$  joining  $u$  and  $v$  (possibly  $\text{length}(Q) = 0$ , i.e.,  $u = v$ ). Now it is easy to verify that the graph induced by the vertices of  $P \cup Q$  contains a hole, a contradiction.

(ii): Assume to the contrary that  $N_A(x)$  and  $N_A(y)$  are not comparable. Let  $z$  and  $t$  be two vertices of  $A$  such that  $z \in N_A(x) - N_A(y)$  and  $t \in N_A(y) - N_A(x)$ . If  $z$  is adjacent to  $t$  then  $x, z, t, y, w$  (where  $w$  is the vertex from condition (i)) induce a  $C_5$ . Hence  $z \not\sim t$ , and by Proposition 1, there is a  $P_4(z, a, b, t)$  in  $A$ . Since by Proposition 3,  $N_A(x)$  and  $N_A(y)$  are cliques, neither  $x$  nor  $y$  can be adjacent to both vertices  $a$  and  $b$ . It follows that the subgraph induced by  $x, z, a, b, t, y, w$  contains a hole, a contradiction.  $\square$

**Proposition 7.** *Let  $A^*$  be a maximal matched co-bipartite subgraph of  $G$  containing  $A$ . Then the following hold:*

- (i) *Every vertex of  $A_6$  is total with respect to  $V(A^*)$ .*
- (ii) *If  $x$  and  $y$  are vertices of  $G \setminus A^*$  with  $x, y \in A_3$ ,  $N_A(x) = \text{left}(A)$  and  $N_A(y) = \text{right}(A)$  then  $x \not\sim y$ .*

**Proof.** (i): Assume to the contrary that for some  $x \in A_6$  and  $y \in V(A^*) - V(A)$ ,  $x \not\sim y$  holds. Assume without loss of generality that  $y \in \text{left}(A^*)$  and let  $z$  be the neighbor of  $y$  in  $\text{right}(A^*)$ . Consider the subgraph  $H$  of  $G$  induced by  $a, b, c, d, y, z$  where  $a, b, c, d$  are four vertices of  $A$  forming a  $C_4$ . Clearly,  $H$  is isomorphic to a  $\overline{C_6}$ . Since  $x$  is total with respect to  $\{a, b, c, d\}$ ,  $x$  will be adjacent to four or five vertices of  $H$  and we obtain a contradiction to Proposition 3.

(ii): First observe that if  $A^* = A$  then  $x \not\sim y$  for otherwise the graph induced by  $V(A) \cup \{x, y\}$  is a matched co-bipartite graph and this contradicts the maximality of  $A^*$ . Thus, we can suppose that  $V(A^*) - V(A) \neq \emptyset$ .

Assume to the contrary that  $x \sim y$  and consider any edge  $zt$  of  $A^* - A$  such that  $z \in \text{left}(A^*)$  and  $t \in \text{right}(A^*)$ . Let  $Q$  be the graph induced by  $z, t$  and four vertices  $a, b, c, d$  forming a  $C_4$  in  $A$  such that  $\{a, b\} \subset \text{left}(A)$  and  $\{c, d\} \subset \text{right}(A)$ . Clearly  $Q$  is isomorphic to a  $\overline{C_6}$ . We shall prove that  $x \sim z$ ,  $y \sim t$ ,  $x \not\sim t$  and  $y \not\sim z$ . Observe first that since  $x$  misses  $c, d$  and  $y$  misses  $a, b$ , we must have that  $x \not\sim t$  and  $y \not\sim z$  for otherwise  $N_Q(x)$  or  $N_Q(y)$  would not be a clique which contradicts Proposition 3.

Let  $Q_2$  ( $Q_3$ , respectively) denote the vertices outside  $Q$  having exactly two neighbors (three neighbors, respectively) in  $Q$ . Now  $x \sim z$  and  $y \sim t$  for otherwise since  $x$  sees  $a$  and  $b$ , and  $y$  sees  $c$  and  $d$ , we would have  $x \in Q_2$  and  $y \in Q_2 \cup Q_3$  or  $x \in Q_2 \cup Q_3$  and  $y \in Q_2$ , and we obtain a contradiction to Proposition 4 or Proposition 5. Hence  $x \textcircled{1} \text{left}(A^*)$ ,  $x \textcircled{0} \text{right}(A^*)$ ,  $y \textcircled{1} \text{right}(A^*)$  and  $y \textcircled{0} \text{left}(A^*)$  and consequently  $V(A^*) \cup \{x, y\}$  induces a graph isomorphic to a matched co-bipartite graph which contradicts to the assumed maximality of  $A^*$ .  $\square$



### 3.2 A Lemma for Atoms of HP-Free Graphs

The subsequent Lemma 1 describes an essential property of HP-free atoms which will lead to a structural description of HP-free graphs.

Let  $G$  be an HP-free graph, let  $A$  be an induced  $\overline{C_6}$  in  $G$  and let  $xy$  be a matching edge of  $A$  with  $x \in \text{left}(A)$  and  $y \in \text{right}(A)$ . We use the following notation:

- $A_2[xy] := \{u \mid u \in A_2, N_A(u) = \{x, y\}\}$
- $A_1[xy] := \{uv \in E \mid u, v \in A_1, N_A(u) = \{x\}, N_A(v) = \{y\}\}$ .

By  $V(A_1[xy])$ , we denote the set of vertices in  $A_1[xy]$ .

**Lemma 1.** *In an HP-free atom,  $A_1[xy] = A_2[xy] = \emptyset$ .*

**Proof.** Assume to the contrary that at least one of the two sets is nonempty. Recall that by Proposition 3 (iv),  $A_6$  is a clique which implies that  $\{x, y\} \cup A_6$  is a clique. Let  $G' := G \setminus (\{x, y\} \cup A_6)$  and  $A' := A \setminus \{x, y\}$ . Clearly the vertices of  $A'$  form a  $C_4$ , say  $C = (a, b, c, d)$  with  $\text{left}(A) = \{x, a, d\}$  and  $\text{right}(A) = \{y, b, c\}$ . Since  $G$  is an atom,  $\{x, y\} \cup A_6$  can not be a clique cutset and consequently,  $G'$  contains a path between some vertex  $x_0 \in A_2[xy] \cup V(A_1[xy])$  and  $x_k \in A'$ . Let  $L = (x_0, x_1, \dots, x_k)$  be such a path of minimum length in  $G'$ . If  $x_0 y_0 \in A_1[xy]$  then we assume without loss of generality that  $x_0 \sim x$  and  $y_0 \sim y$ .

**Claim 1.**  $\text{length}(L) > 2$ .

*Proof of Claim 1.* Assume not - then  $L = (x_0, x_1, x_2)$  with  $x_2 \in A'$ .

Assume first that  $x_0 \in A_2[xy]$ . Since by Proposition 3,  $N_A(x_1)$  is a clique (recall that  $x_1 \notin A_6$ ) and  $N_A(x_1) \cap \{a, b, c, d\} \neq \emptyset$ , if  $x_1 \in A_1 \cup A_3$  then  $N_A(x_0)$  is not comparable with  $N_A(x_1)$  which contradicts Proposition 4 and if  $x_1 \in A_2$ ,  $N_A(x_0) \cup N_A(x_1)$  is not a clique which contradicts Proposition 5.

Assume now that  $x_0 \in V(A_1[xy])$  (recall that we assumed  $x_0 \sim x$ ). By Proposition 2 and Proposition 4 we deduce that  $N_A(x_1) \subseteq \{x, a, d\}$  and that  $y_0 \not\sim x_1$ . Let  $u$  be a neighbor of  $x_1$  in  $\{a, d\}$  and  $v$  the vertex of  $\{b, c\}$  adjacent to  $u$ . Then  $x_0, x_1, u, v, y, y_0$  induce a  $C_6$ , a contradiction which shows Claim 1.  $\diamond$

Since  $\text{length}(L)$  is assumed to be minimum, none of  $x_1, \dots, x_{k-2}$  can be in  $A_2 \cup A_3 \cup V(A_1[xy]) \cup A_2[xy]$ . It follows that if a vertex  $x_i \in \{x_1, \dots, x_{k-2}\}$  belongs to  $D_1$  then  $x_i \in A_1 - V(A_1[xy])$ . Let

$$Q := \{x_1, \dots, x_{k-2}\} \cap (A_1 - V(A_1[xy])).$$

**Claim 2.** *If  $x_0 \in A_2[xy]$  then  $Q \neq \emptyset$ .*

*Proof of Claim 2.* Assume  $Q = \emptyset$ ; then none of  $x_1, \dots, x_{k-2}$  belongs to  $D_1$  and consequently by Proposition 6,  $N_A(x_{k-1})$  and  $N_A(x_0) = \{x, y\}$  must be comparable. By Proposition 3,  $N_A(x_{k-1})$  must be a clique (recall that  $x_k \in \{a, b, c, d\}$ , and since the path in  $G'$  contains no vertex from  $A_6$ , we have  $x_{k-1} \notin A_6$ ). Thus we obtain a contradiction which shows Claim 2.  $\diamond$

**Claim 3.** *If  $Q \neq \emptyset$  then either  $N_A(Q) = \{x\}$  or  $N_A(Q) = \{y\}$ .*

*Proof of Claim 3.* Assume not; then there are two vertices  $x_i$  and  $x_j$  in  $Q$ ,  $1 \leq i < j \leq k-2$ , such that  $N_A(x_i) \neq N_A(x_j)$  and for all  $k$ ,  $i < k < j$ ,  $x_k \notin D_1$ . Observe that  $j > i + 1$  for otherwise  $x_i$  would be adjacent to  $x_j$  and consequently  $x_i$  and  $x_j$  would belong to  $V(A_1[x, y])$ , a contradiction. Now  $N_A(x_i)$  and  $N_A(x_j)$  are not comparable - a contradiction to Proposition 6 which shows Claim 3.  $\diamond$

**Claim 4.** *If  $Q \neq \emptyset$  then  $N_A(Q) = \{x\}$  implies that  $N(x_{k-1}) \subseteq \text{left}(A)$  and  $N_A(Q) = \{y\}$  implies that  $N(x_{k-1}) \subseteq \text{right}(A)$ .*

*Proof of Claim 4.* Let  $x_s$ ,  $1 \leq s \leq k-2$ , be a vertex of path  $L$  with  $x_s \in Q$  such that  $s$  is maximum with respect to these properties.

Assume first that  $x_{k-1} \in A_1$ . Then  $x_s \sim x_{k-1}$  for otherwise, by Proposition 6,  $N_A(x_{k-1})$  must be comparable with  $N_A(x_s)$  and we obtain a contradiction to the fact that  $x_{k-1}$  has a neighbor in  $\{a, b, c, d\}$ . Proposition 2 implies that  $N_A(x_{k-1}) \sim N_A(x_s)$  and consequently  $N_A(x_{k-1})$  is contained either in  $\{a, d\} \subset \text{left}(A)$  if  $N_A(x_s) = \{x\}$  or in  $\{b, c\} \subset \text{right}(A)$  if  $N_A(x_s) = \{y\}$ .

Now assume that  $x_{k-1} \in A_2 \cup A_3$ . Then Proposition 4 and Proposition 6 imply that  $N_A(x_{k-1})$  and  $N_A(x_s)$  must be comparable. Claim 4 follows from the fact that  $N_A(x_{k-1})$  is a clique and at least one of the vertices of  $\{a, b, c, d\}$  belongs to  $N_A(x_{k-1})$ .  $\diamond$

**Claim 5.** *For  $x_0 \in V(A_1[xy])$ , the following hold:*

- (i) *If  $Q \neq \emptyset$  then  $N_A(Q) = \{x\}$ .*
- (ii)  *$N_A(x_{k-1}) \subseteq \text{left}(A)$ .*

*Proof of Claim 5.* (i): Recall that for  $x_0 \in V(A_1[xy])$ , we assumed that  $N_A(x_0) = \{x\}$ . Let  $x_i$  be a vertex such that  $x_i \in Q$  and  $i$  is as small as possible. Recall that by Claim 3, either  $N_A(Q) = \{x\}$  or  $N_A(Q) = \{y\}$  holds.

If  $i = 1$  and  $N_A(Q) = \{y\}$  then  $x_1 \in V(A_1[xy])$  since  $x_1 \sim x_0$  - a contradiction to the fact that every vertex of  $Q$  belongs to  $A_1 - V(A_1[xy])$ . Thus,  $N_A(x_1) = \{x\}$  and also  $N_A(Q) = \{x\}$ .

If  $i > 1$  then  $x_1 \in D_2$  and by Proposition 6 we obtain that  $i = 2$  and  $N_A(x_2) = \{x\}$ . Then by Claim 2 we obtain that  $N_A(Q) = \{x\}$  as claimed.

(ii): If  $Q \neq \emptyset$  then  $N_A(x_{k-1}) \subseteq \text{left}(A)$  follows by the fact that  $N_A(Q) = \{x\}$  and Claim 4. In the other case, if  $Q = \emptyset$  then no vertex of  $\{x_1, \dots, x_{k-2}\}$  is in  $D_1$ . Proposition 6 implies that  $N_A(x_{k-1})$  and  $N_A(x_0)$  must be comparable, and since by assumption  $N_A(x_0) = \{x\}$  and  $N_A(x_{k-1})$  is a clique, we obtain Claim 5.  $\diamond$

Let  $u \in \{a, d\}$  be a neighbor of  $x_{k-1}$  and let  $v$  be the neighbor of  $u$  in  $\text{right}(A)$  which clearly is different from the vertex  $y$ . If  $x_0 \in A_2[xy]$  then by Claim 2,  $Q \neq \emptyset$  and by Claim 3,  $N_A(Q) = \{x\}$  or  $N_A(Q) = \{y\}$ . Assume without loss of generality that  $N_A(Q) = \{x\}$ ; then by Claim 4, we have  $N(x_{k-1}) \subseteq \text{left}(A)$ . Then the subgraph induced by  $x_0, \dots, x_{k-1}, u, v, y$  is a hole, a contradiction. Hence  $x_0 \in V(A_1[xy])$ . By Claim 5, if  $Q \neq \emptyset$  then  $N_A(Q) = \{x\}$ . It follows that the subgraph induced by  $x_0, \dots, x_{k-1}, u, v, y, y_0$  is a hole, a contradiction which shows Lemma 1.  $\square$



## 4 Structure of (Hole,Paraglider)-Free and (Hole,Diamond)-Free Atoms

Recall that HP-free (HD-free, respectively) denotes hole- and paraglider-free (hole- and diamond-free, respectively).

**Theorem 2.** *If  $G$  is an HP-free atom containing an induced  $\overline{C_6}$   $A$ , and  $A_6$  denotes the set of vertices which are universal for  $A$  then  $G \setminus A_6$  is a matched co-bipartite graph.*

**Proof.** Assume the contrary; let  $G' := G \setminus A_6$  and let  $A^*$  be a maximal matched co-bipartite subgraph in  $G'$  containing  $A$ . Let  $W := V(G') - V(A^*)$ ; by assumption,  $W \neq \emptyset$ . We define a partition  $\pi(W)$  of the vertices of  $W$  according to their distance from  $A^*$ :  $W = W_1 \cup \dots \cup W_k$  where  $W_i := \{x \in W \mid d(x, A^*) = i\}$ ,  $i = 1, \dots, k$ . Thus,  $W_1 = (W \cap (A_1 \cup A_2 \cup A_3)) \cup (W \cap D_2^*)$  where  $D_2^*$  denotes the set of vertices which are in distance two from  $A$  and which see a vertex in  $A^*$ . The vertices in  $W_2$  have distance at least two from  $A$ .

**Claim 6.** *No vertex in  $W_1$  has neighbors in both  $\text{left}(A^*)$  and  $\text{right}(A^*)$ .*

*Proof of Claim 6.* Assume to the contrary that for some  $x \in W_1$ , there are  $y$  and  $z$  with  $y \in \text{left}(A^*)$  and  $z \in \text{right}(A^*)$  such that  $x \sim y$  and  $x \sim z$ .

Suppose first that  $y \sim z$ . Consider the graph  $Q$  induced by  $y, z$  and four vertices  $a, b, c, d$  of  $A$  forming a  $C_4$  such that  $\{y, z\} \cap \{a, b, c, d\} = \emptyset$ . Clearly  $Q$  is isomorphic to a  $\overline{C_6}$ . Then since by Lemma 1,  $Q_2[yz] = \emptyset$  (where as before,  $Q_2[yz]$  denotes the vertices outside  $Q$  seeing exactly  $y$  and  $z$  in  $Q$ ),  $x$  can not belong to  $D_2(A)$  and consequently  $N(x) \cap \{a, b, c, d\} \neq \emptyset$ , that is,  $x \in A_1 \cup A_2 \cup A_3$ . Since by Proposition 3,  $N_Q(x)$  is a clique and by assumption  $x$  sees both  $y$  and  $z$ , we obtain a contradiction.

Now suppose that  $y \not\sim z$  and consider the graph  $H$  induced by  $y, z, y_1, z_1, a, b$  where  $y_1$  is the neighbor of  $y$  in  $\text{right}(A^*)$ ,  $z_1$  is the neighbor of  $z$  in  $\text{left}(A^*)$ ,  $ab$  is any edge of  $A$  such that  $a \in \text{left}(A)$ ,  $b \in \text{right}(A)$  and  $\{a, b\} \cap \{y, y_1, z, z_1\} = \emptyset$ . Clearly  $H$  is isomorphic to a  $\overline{C_6}$ . Since by assumption  $x$  sees both  $y$  and  $z$ ,  $N_H(x)$  is not a clique which by Proposition 3 (v) implies that  $x$  sees all vertices of  $H$  and thus also  $x \sim a$  and  $x \sim b$  with  $a \in \text{left}(A)$  and  $b \in \text{right}(A)$ . Since by Proposition 3,  $x \notin A_3$ , by Lemma 1,  $x \notin A_2[a, b]$  and by assumption,  $x \notin A_6$ , we obtain a contradiction.  $\diamond$

We define now the following sets:

$$\text{left}(W_1) := \{x \in W_1 \mid N_{A^*}(x) \subseteq \text{left}(A^*)\} \text{ and}$$

$$\text{right}(W_1) := \{x \in W_1 \mid N_{A^*}(x) \subseteq \text{right}(A^*)\}.$$

By Claim 6,  $\text{left}(W_1) \cap \text{right}(W_1) = \emptyset$ . Thus  $W_1 = \text{left}(W_1) \cup \text{right}(W_1)$  is a partition of  $W_1$ .

**Claim 7.** *There is no edge between  $\text{left}(W_1)$  and  $\text{right}(W_1)$ .*

*Proof of Claim 7.*

Assume to the contrary that  $x \sim y$  for some  $x \in \text{left}(W_1)$  and  $y \in \text{right}(W_1)$ . Recall that  $D_1$  denotes the vertices in distance one to  $A$ . We first show:

$$x \text{ and } y \text{ can not be both in } D_1. \tag{1}$$

Assume to the contrary that  $x, y \in D_1$ . Then by Proposition 7 (ii),  $x, y \in A_3$  is impossible. Suppose without loss of generality that  $x \notin A_3$ , i.e.,  $x \in A_1 \cup A_2$  and  $y \in A_1 \cup A_2 \cup A_3$ . If  $x \in A_1$  and  $y \in A_2 \cup A_3$  or  $x \in A_2$  and  $y \in A_1 \cup A_3$ , Proposition 4 implies that  $N_A(x)$  and  $N_A(y)$  are comparable, and if  $x, y \in A_2$ , Proposition 5 implies that  $N_A(x) \cup N_A(y)$  is a clique. But since  $x \in \text{left}(W_1)$  and  $y \in \text{right}(W_1)$ , none of these cases can occur. It follows that  $x, y \in A_1$ . However, by Lemma 1, such a pair of adjacent vertices can not exist, a contradiction.  $\diamond$

It follows that at least one of  $x$  or  $y$  is in  $D_2$ . Assume that  $x \in D_2$  and let  $u$  be a neighbor of  $x$  in  $D_1$ . Suppose first that also  $y \in D_2$  and let  $v$  be a neighbor of  $y$  in  $D_1$ . Obviously  $u \in \text{left}(W_1)$  and  $v \in \text{right}(W_1)$ . Since by assumption  $x, y \in D_2$ , Proposition 6 (i) implies that  $u \sim v$  and we obtain a contradiction with (1). Consequently,  $y \in D_1$ . Since  $N_A(u)$  and  $N_A(y)$  are not comparable, Proposition 6 (ii) implies that  $u \sim y$  and again we obtain a contradiction with (1). This shows Claim 7.  $\diamond$

For the partition  $\pi(W) = \{W_1, \dots, W_k\}$ ,  $k \geq 1$ , define the following sets for every  $i \in \{2, \dots, k\}$ :

$$\text{left}(W_i) := \{x \in W_i \mid \exists y \in \text{left}(W_{i-1}) \text{ such that } x \sim y\} \text{ and}$$

$$\text{right}(W_i) := \{x \in W_i \mid \exists y \in \text{right}(W_{i-1}) \text{ such that } x \sim y\}.$$

**Claim 8.**  $(\text{left}(W_1) \cup \dots \cup \text{left}(W_k)) \cap (\text{right}(W_1) \cup \dots \cup \text{right}(W_k)) = \emptyset$  and  $(\text{left}(W_1) \cup \dots \cup \text{left}(W_k)) \textcircled{\cap} (\text{right}(W_1) \cup \dots \cup \text{right}(W_k))$ .

*Proof of Claim 8.* We shall prove the claim by induction on  $k$ . By Claims 6 and 7, the result is true for  $k = 1$ . By the induction hypothesis the result is true for  $k < s$ ,  $s > 1$ . Assume to the contrary that the result is false for  $W_s \in \pi(W)$ . Then there must be a chordless path  $L_1 = (x_1, \dots, x_{s-1}, x, y_{s-1}, \dots, y_1)$  or a chordless path  $L_2 = (x_1, \dots, x_{s-1}, x, y, y_{s-1}, \dots, y_1)$  such that  $x_i \in \text{left}(W_i)$ ,  $y_i \in \text{right}(W_i)$ ,  $i \in \{1, \dots, s-1\}$  and  $x, y \in W_s$ . By the induction hypothesis there is no edge between  $\{x_1, \dots, x_{s-1}\}$  and  $\{y_1, \dots, y_{s-1}\}$ . Let  $L = (x_1, z_1, \dots, z_r, y_1)$ ,  $r \geq 2$ , be a chordless path joining  $x_1$  and  $y_1$  such that  $z_i \in A^*$ ,  $i \in \{1, \dots, r\}$ , which clearly exists. It is easy to see that the graph induced by the vertices of  $L_1$  and  $L$  or by the vertices of  $L_2$  and  $L$  is isomorphic to a hole - a contradiction. This shows Claim 8.  $\diamond$

Let

$$\text{left}(W) := (\text{left}(W_1) \cup \dots \cup \text{left}(W_k)) \text{ and}$$

$$\text{right}(W) := (\text{right}(W_1) \cup \dots \cup \text{right}(W_k)).$$

By Claim 8,  $\text{left}(W)$  and  $\text{right}(W)$  form a partition of  $W$ .

**Claim 9.**  $\text{left}(W) \textcircled{\cap} \text{right}(W) \cup \text{right}(A^*)$  and  $\text{right}(W) \textcircled{\cap} \text{left}(W) \cup \text{left}(A^*)$ .

*Proof of Claim 9.* Indeed, by Claim 8, we have that  $\text{left}(W) \textcircled{\cap} \text{right}(W)$ . By Claim 6, we have that  $\text{left}(W_1) \textcircled{\cap} \text{right}(A^*)$  and  $\text{right}(W_1) \textcircled{\cap} \text{left}(A^*)$ , and by the construction of  $W_2, \dots, W_k$  we have that  $(W_2 \cup \dots \cup W_k) \textcircled{\cap} V(A^*)$ . This shows Claim 9.  $\diamond$

Since by assumption  $G' = G \setminus A_6$  is not isomorphic to a matched co-bipartite graph, we must have that  $W \neq \emptyset$ . Assume without loss of generality that  $\text{left}(W) \neq \emptyset$ . Then

since by Proposition 7 (i),  $A_6 \cup \text{left}(A^*)$  is a clique and since by Claim 9, there is no edge between  $\text{left}(W)$  and  $\text{right}(W) \cup \text{right}(A^*)$ ,  $A_6 \cup \text{left}(A^*)$  would be a clique cutset in  $G$  which contradicts our assumption that  $G$  is an atom. This finishes the proof of Theorem 2.  $\square$

**Corollary 2.** *Let  $G$  be a (hole,paraglider)-free graph.*

- (i) *If  $G$  is  $\overline{C_6}$ -free then  $G$  is weakly chordal.*
- (ii) *If  $G$  is an atom containing an induced  $\overline{C_6}$  then  $G$  is the join of a matched co-bipartite graph and a clique.*

**Proof.** (i): Recall that HP-free graphs are  $\overline{C_k}$ -free for  $k \geq 7$ .

(ii): Indeed by Theorem 2, for a  $\overline{C_6}$   $A$  in  $G$ ,  $G' = G \setminus A_6$  is a matched co-bipartite graph. By Proposition 7,  $A_6 \textcircled{1} V(G')$ , and by Proposition 3,  $A_6$  is a clique.  $\square$

Since by Proposition 3 (iv), in (hole,diamond)-free graphs  $A_6 = \emptyset$ , we have:

**Corollary 3.** *Let  $G$  be a (hole,diamond)-free graph.*

- (i) *If  $G$  is  $\overline{C_6}$ -free then  $G$  is weakly chordal.*
- (ii) *If  $G$  is an atom containing an induced  $\overline{C_6}$  then  $G$  is a matched co-bipartite graph.*

## 5 Algorithmic Consequences

In [32], for various problems such as Minimum Fill-in, Maximum Independent Set, Maximum Clique and Coloring, it is shown that whenever these problems are efficiently solvable on the atoms of a graph class, they are efficiently solvable on the graphs of the class. For perfect graphs, Maximum Independent Set, Maximum Clique and Coloring are known to be solvable in polynomial time [23, 24] using the ellipsoid method (but from a practical point of view, this is not an efficient solution of the problems).

(Hole,paraglider)-free graphs are perfect as the Strong Perfect Graph Theorem implies (a more direct way can use Theorem 2 and Corollary 2 and the fact that a graph is perfect if its atoms are perfect).

The clique separator approach gives direct combinatorial algorithms for the problems mentioned above:

Recognition of weakly chordal graphs can be done in  $\mathcal{O}(m^2)$  [2, 27], and recognition of matched co-bipartite graphs can be easily done in linear time. Thus, given an input graph, determine its atoms and check whether they are either weakly chordal or are the join of a clique and a matched co-bipartite graph. If not then the input graph is not (hole,paraglider)-free. Otherwise solve the problems on the atoms and finally combine the solutions as described in [32].

For matched co-bipartite graphs, MWIS is trivial. A first polynomial time algorithm for weakly chordal graphs is given in [26], and in [31], MWIS is solved in time  $\mathcal{O}(n^4)$  for weakly chordal graphs. Thus, the time bound for MWIS on HP-free graphs is roughly  $\mathcal{O}(n^6)$ : Determine whether the input graph is weakly chordal. If yes, use the algorithm for weakly chordal graphs. If not, check whether all prime atoms are matched co-bipartite,

and if yes, then use the trivial algorithm for these graphs. If not, the input graph is not HP-free.

For Maximum Clique and Coloring one can proceed in a similar way. For Maximum Clique on diamond-free graphs, however, there is a more direct way to solve the problem efficiently by switching to the complement graph and the complement problem MWIS: If  $G$  is gem-free (see Figure 1 for gem) then  $\overline{G}$  has the property that for every vertex, its antineighborhood is  $P_4$ -free, i.e., a cograph. This means that one can solve the MWIS problem for such graphs in time  $\mathcal{O}(nm)$  in the obvious way.

In [5], a  $\mathcal{O}(n^6)$  algorithm is given for Minimum Fill-In on weakly chordal graphs. Minimum Fill-In on matched co-bipartite graphs is efficiently solvable in the obvious way.

The Maximum Weight Induced Matching (MWIM) problem is another example of a problem which can be added to the list of problems above: A set  $M$  of edges is an *induced matching* in  $G$  if the pairwise distance of the edges in  $M$  is at least two in  $G$ . The MWIM problem asks for an induced matching of maximum weight. In [16], it is shown that for a hereditary class  $\mathcal{C}$  of graphs, MWIM is solvable in polynomial time if MWIM is solvable in polynomial time on the atoms of  $\mathcal{C}$ . This can be applied to (hole,paraglider)-free graphs since for weakly chordal graphs, a polynomial time solution is given in [18], and obviously, matched co-bipartite graphs are  $3K_2$ -free, which means that in such graphs (and in the join of a matched co-bipartite graph and a clique) one has to check only pairs of edges.

## 6 Conclusion

In this paper we have described the structure of (hole, paraglider)-free atoms (of (hole, diamond)-free atoms, respectively) and some algorithmic consequences. In a forthcoming paper [3] we will analyze the structure of (hole,diamond)-free graphs and its algorithmic consequences in more detail; in particular, we show that weakly chordal diamond-free atoms are either cliques or chordal bipartite.

There are various other aspects and papers which are related of our work as described subsequently:

### 6.1 Related results for subclasses of $P_5$ -free graphs

In [1], Alekseev showed that  $P_5$ - and paraglider-free atoms are  $3K_2$ -free which leads to a polynomial time algorithm for the MWIS problem since  $3K_2$ -free graphs contain at most  $\mathcal{O}(n^4)$  inclusion-maximal independent sets. In [11], we improved this result by generalizing the forbidden paraglider subgraph. In [8], we give a more detailed structural analysis of  $P_5$ - and paraglider-free atoms. In [15], we describe the structure of prime  $P_5$ - and co-chair-free graphs and give algorithmic applications. The complexity of the MWIS problem for  $P_5$ -free graphs is an open problem. It is also open for  $(P_5, C_5)$ -free graphs; such graphs are hole-free. Thus, it is interesting to study subclasses of  $P_5$ -free graphs (subclasses of  $(P_5, C_5)$ -free graphs, respectively).

### 6.2 Clique-width

In [6], we describe the simple structure of  $(P_5, \text{diamond})$ -free graphs; such graphs can contain  $C_5$  and thus,  $P_5$ - and diamond-free graphs are in general not perfect and incomparable

with (hole,diamond)-free graphs.  $(P_5,\text{diamond})$ -free graphs have bounded clique-width - see e.g. [20] for the notion and algorithmic implications of bounded clique-width which has tremendous consequences for efficiently solving hard problems on such graph classes. For the more general class of  $(P_5,\text{gem})$ -free graphs, the situation is similar: By the Strong Perfect Graph Theorem, (hole,gem)-free graphs are perfect since antiholes with at least seven vertices contain gem. The structure of  $(P_5,\text{gem})$ -free graphs and some algorithmic applications were described in [4, 9]. In [12], it was shown that  $(P_5,\text{gem})$ -free graphs have bounded clique-width.

The clique-width of (hole,diamond)-free graphs, however, is unbounded since e.g. the subclass of chordal bipartite graphs (which are the (hole, triangle)-free graphs), has unbounded clique-width [14]. This illustrates that corresponding subclasses of hole-free graphs are more interesting than those of  $P_5$ -free graphs.

### 6.3 Open problems

It would be interesting to describe the structure of (hole,gem)-free graphs. In particular, how can one avoid to use the Strong Perfect Graph Theorem for showing that (hole,gem)-free graphs are perfect?

In [7], we give a polynomial time algorithm for the MWIS problem on hole- and co-chair-free graphs. It would be interesting to obtain better structural results on these graphs.

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